review

Wednesday, May 6, 2026 8:05 AM

tip: practice manipulating matrices "by hand" if you've been using technology

$\downarrow$ addition, multiplication, rref, determinants

tip: Wronskian factor

given $\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}, \begin{bmatrix} e^{-2t} \\ 5e^{-2t} \end{bmatrix}$

can factor $e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

then $W = \underbrace{e^{2t} e^{-2t}}_{1} \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$

$(5-1) = 4 \neq 0$

given $x_1' = -3x_1 + 4x_2$
$x_2' = 6x_1 - 5x_2$

$\vec{x}' = \begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix} \vec{x} \longrightarrow \det \begin{bmatrix} -3-\lambda & 4 \\ 6 & -5-\lambda \end{bmatrix} = 0$

$(-3-\lambda)(-5-\lambda) - 24 = 0$

$\lambda^2 + 8\lambda + 15 - 24 = 0$

$\lambda^2 + 8\lambda - 9 = 0$

$(\lambda+9)(\lambda-1) = 0$

$\lambda_1 = -9 \quad \lambda_2 = 1$

when $\lambda = -9$ $\begin{bmatrix} 6 & 4 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$6v_1 + 4v_2 = 0$

$6v_1 = -4v_2$

$v_1 = -\frac{2}{3}v_2$

$v_2 = 1v_2$

$EV_{\lambda=-9}: \begin{bmatrix} -2/3 \\ 1 \end{bmatrix} 3 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$\vec{x}_1(t) = \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-9t}$

when $\lambda = 1$ $\begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{x}_1(t) = \begin{bmatrix} \\ 3 \end{bmatrix} e$

when $\lambda = 1$ $\quad \begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-4v_1 + 4v_2 = 0$

$v_1 = 1v_2$

$v_2 = 1v_2$

$EV_{\lambda=1}: \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{x}_2(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$

$$\vec{x}(t) = c_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-9t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$$

$$\updownarrow$$

$$x_1(t) = -2c_1 e^{-9t} + c_2 e^{t}$$
$$x_2(t) = 3c_1 e^{-9t} + c_2 e^{t}$$

be able to express sol'n in either form

multiplicity

tip: don't assume an eigenvalue is defective just bc $k > 1$

look @ 1ST example on 20-Apr L18

eigenvalue set was complete bc v_3 was arbitrary

higher level than you'll see on exam:

given $\vec{x}' = \begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix} \vec{x} \longrightarrow \det \begin{bmatrix} -\lambda & 1 & 2 \\ -5 & -3-\lambda & -7 \\ 1 & 0 & -\lambda \end{bmatrix} = 0$

$$1 \det \begin{bmatrix} 1 & 2 \\ -3-\lambda & -7 \end{bmatrix} - 0 - \lambda \det \begin{bmatrix} -\lambda & 1 \\ -5 & -3-\lambda \end{bmatrix} = 0$$

$$-7 - 2(-3-\lambda) \quad -\lambda\left[-\lambda(-3-\lambda) + 5\right] = 0$$

$$\underbrace{-7 + 6} + 2\lambda \quad -\lambda\left[+3\lambda + \lambda^2 + 5\right] = 0$$

$$-1 + 2\lambda \quad -3\lambda^2 - \lambda^3 - 5\lambda = 0$$

$$-\lambda^3 - 3\lambda^2 - 3\lambda - 1 = 0$$

$$-(1\lambda^3 + 3\lambda^2 + 3\lambda + 1) = 0$$

$$-(\lambda+1)^3 = 0$$

$\lambda = -1 \quad k = 3$

when $\lambda = -1$

$$\begin{bmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_1 + v_3 = 0$

$v_3 = -1v_1$

$v_1 + v_2 + 2v_3 = 0$

$v_1 + v_2 - 2v_1 = 0$

$v_2 - v_1 = 0$

$v_2 = 1v_1$

$v_1 = 1v_1$

EV: $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\lambda = -1$ defective

then $(A-\lambda I_3)^3 v_3 = \vec{0} \longrightarrow \overset{(A+I_3)}{(A-\lambda I_3)} \vec{v}_3 = \vec{v}_2 \longrightarrow (A-\lambda I_3)\vec{v}_2 = \vec{v}_1$

1st find $\vec{v}_3$

$$\overset{A^2}{\begin{bmatrix} -2 & -1 & -3 \\ -2 & -1 & -3 \\ 2 & 1 & 3 \end{bmatrix}} \begin{bmatrix} A \end{bmatrix} \xrightarrow{A^3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

any non-zero vector works

chose $v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$(A-\lambda I_3)\vec{v}_3 = \vec{v}_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{v}_2 = \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix}$$

$(A-\lambda I_3)\vec{v}_2 = \vec{v}_1$

$$\begin{bmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix} = \vec{v}_1 \longrightarrow \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$$

$\longleftarrow$ scalar of EV from earlier (check) develop sol'ns $\lambda = -1 \quad k=3$

$$\vec{x}_1(t) = v_1 e^{\lambda t} = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} e^{-t}$$

$$\vec{x}_2(t) = (v_1 t + v_2) e^{\lambda t} = \left(\begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} t + \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix} \right) e^{-t}$$

$$\vec{x}_3(t) = \left(\underbrace{\frac{1}{2} \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}}_{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}} t^2 + \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) e^{-t}$$

$$\overset{A}{\begin{bmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1-5+2 & & -5+4+0 \\ & & \\ & & \end{bmatrix}$$

Fundamental Matrix – using this to find sol'ns

given $\vec{x}' = A\vec{x}$ when $\vec{x}(0) = \vec{x}_0$

then $\vec{x}(t) = \phi(t)\,\phi^{-1}(0)\,\vec{x}(0)$

Know how to find an inverse matrix $\quad \phi = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \phi^{-1} \frac{1}{\det(\phi)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

given $\vec{x}' = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \vec{x} \qquad \vec{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

char eqn $\quad \det \begin{bmatrix} 2-\lambda & -1 \\ -4 & 2-\lambda \end{bmatrix} = 0$

$$\lambda(\lambda - 4) = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = 4$$

when $\lambda = 0$

$$\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

when $\lambda = 4 \qquad EV_{\lambda=4}: \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} -4 & 2 \end{bmatrix}\begin{bmatrix} v_2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$2v_1 - v_2 = 0$

$v_2 = 2v_1$ $\quad EV_{\lambda=0} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$v_1 = v_1$

$\vec{x}_1(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{x}_2(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{4t} = \begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix}$

$$\phi(t) = \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{bmatrix}$$

$$\det\phi = -2e^{4t} - 2e^{4t} = -4e^{4t}$$

$$\phi^{-1}(t) = -\frac{1}{4e^{4t}} \begin{bmatrix} -2e^{4t} & -e^{4t} \\ -2 & 1 \end{bmatrix}$$

$$\phi^{-1}(0) = -\frac{1}{4} \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix}$$

$$\vec{x}(t) = \phi(t)\,\phi^{-1}(0)\,\vec{x}(0)$$

$$= \underbrace{-\frac{1}{4}\begin{bmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{bmatrix} \underbrace{\begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{2\times 1}}$$

$$= \vec{x}(t) = \frac{1}{4}\begin{bmatrix} 3+5e^{4t} \\ 6-10e^{4t} \end{bmatrix}$$

Matrix Exponentials

given $\vec{x}' = \underbrace{\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}}_{A} \vec{x} \quad \vec{x}(0) = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ find e^{At}

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}$$

$$= 2I + B$$

$$B^2 = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

nilpotent for $n \geq 2$

then $e^{At} = e^{(2I+B)t}$

$$= \underbrace{e^{2It}}_{e^{2t}\cdot I} e^{Bt}$$

$$e^{2t}\cdot I\, \underbrace{e^{Bt}}_{\downarrow \text{ expand}}$$

$$\underbrace{e^{2t} I}\Big(\overset{n=0}{I} + \overset{n=1}{Bt} \Big)$$

$$\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 5t \\ 0 & 0 \end{bmatrix} \right)$$

$$\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix} \underbrace{\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 5t \\ 0 & 0 \end{bmatrix} \right)}$$

$$\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 5t \\ 0 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{2t} & 5te^{2t} \\ 0 & e^{2t} \end{bmatrix}$$ since $\vec{x}(t) = e^{At}\vec{x}_0$

$$= \begin{bmatrix} e^{2t} & 5te^{2t} \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} 4e^{2t} + 35te^{2t} \\ 7e^{2t} \end{bmatrix}$$

review separable eqns

use substitution when DE is not
- separable
- linear

ex. $y'' + 4y = 0$ ← temptation to use 'reducible' technique
better to use char eqn

$y = y_c + y_p$ ← review